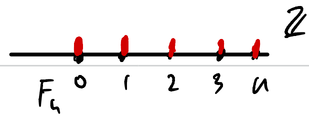
Ergodic Theory and Measured Group Theory
Lecture 17

Proof of Furstenberg Correspondence (continued).

$\mu_n := \frac{1}{|F_n|} \sum_{\gamma \in F_n} \delta_{\gamma \cdot \mathbb{1}_A}$.

$\mu_n(\tilde{B}) := \frac{1}{|F_n|} \left| \{ \gamma \in F_n : \gamma \cdot \mathbb{1}_A \in \tilde{B} \} \right|$, so $\tilde{B} \subseteq X$



$$\mu_n(\tilde{A}) = \frac{1}{|F_n|} \left| \{ \gamma \in F_n : \gamma \cdot \mathbb{1}_A \in \tilde{A} \} \right| = \frac{1}{|F_n|} \left| \{ \gamma \in F_n : \gamma \in A \} \right|$$
$$= \frac{|A \cap F_n|}{|F_n|}.$$

By the Banach-Alaoglu theorem, the space of prob. meas. on X is weak*-compact. Thus, after passing to a subsequence (which we assume WLOG), $\exists$ weak*-limit $\lim_{n\to\infty} \mu_n = \mu$.

μ is invariant because each μ_n is almost invariant (with some ε_n error) and $\varepsilon_n \to 0$ due to the Følner condition. Thus $\Gamma \curvearrowright (2^\Gamma, \mu)$ is a pmp action.

Moreover, $\bar{d}(A) = \lim_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} = \lim_{n\to\infty} \mu_n(\tilde{A}) = \mu(\tilde{A})$.

$$\bar{d}(A\cap g_1^{-1}A\cap\dots\cap g_k^{-1}A) \geq \liminf_{n\to\infty} \frac{|A\cap g_1^{-1}A\cap\dots\cap g_k^{-1}A \cap F_n|}{|F_n|} = (*)$$

$$A\cap g_1^{-1}A\cap\dots\cap g_k^{-1}\cap F_n = \{\gamma\in F_n : \forall i\leq k \ \ \gamma\in g_i^{-1}A\}$$

$$= \{\gamma\in F_n : \forall i\leq k \ \ \gamma\cdot \mathbb{1}_A \in g^{-1}\tilde{A}\}$$

$g_0 := 1_\Gamma$

$$= \{\gamma\in F_n : \gamma\cdot \mathbb{1}_A \in \tilde{A}\cap g_1^{-1}\tilde{A}\cap\dots\cap g_k^{-1}\tilde{A}\}$$

$$= |F_n|\cdot \mu_n(\tilde{A}\cap g_1^{-1}\tilde{A}\cap\dots\cap g_k^{-1}\tilde{A}),$$

so
$$(*) = \liminf_{n\to\infty} \mu_n(\tilde{A}\cap g_1^{-1}\tilde{A}\cap\dots\cap g_k^{-1}\tilde{A})$$

$$= \lim_{n\to\infty} \mu_n(\tilde{A}\cap g_1^{-1}\tilde{A}\cap\dots\cap g_k^{-1}\tilde{A}) = \mu(\quad).$$

$\square$

This shows that Multiple Recurrence $\Rightarrow$ Szemeredi's theorem.

<u>Furstenberg Multiple Recurrence Theorem</u> (1977). For any pmp $\mathbb{Z} \curvearrowright (X,\mu)$, any $A\subseteq X$ of positive measure, $\forall k$ $\exists n$ s.t.

$$\mu(A\cap T^{-n}A\cap\dots\cap T^{-kn}) > 0.$$

In fact, $\forall f\in L^\infty(X,\mu)$, $\forall k$, $f\geq 0$, $\int f\,d\mu > 0$,

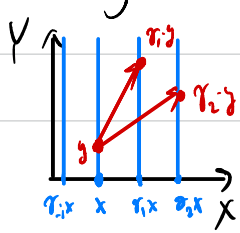
$$\liminf_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} \int f\cdot(T^n f)\cdot\ldots\cdot(T^{kn}f)\,d\mu > 0. \qquad \text{(MR)}$$

We call a pmp action $\mathbb{Z} \curvearrowright (X,\mu)$ Multiple Recurrence action (MR) if the (MR) property above holds for it.

We already "saw" that weakly mixing and compact actions are (MR). The way Furstenberg proves (MR) for all actions is by showing that every action is "built" out of weakly mixing and compact actions. This "built" is by extensions.

Def. For any semigroup action $\Gamma \curvearrowright^{\alpha} (X,\mu)$, another action $\Gamma \curvearrowright^{\beta} (Y,\nu)$ is called a factor of α (or α is called the extension of β) if there is a μ-homomorphism/factor map $\pi : X \to Y$, i.e. $\exists$ measurable $\pi : X \to Y$ that preserve the measure, i.e. $\pi^* \mu = \nu$ (hence onto a.e.), and π is equivariant, i.e. $\forall \gamma \in \Gamma$, $\pi \circ \gamma_\alpha = \gamma_\beta \circ \pi$ a.e.

Example. Products: let $\Gamma \curvearrowright^{\alpha} (X,\mu)$ and $\Gamma \curvearrowright^{\beta} (Y,\nu)$, then take the product action $\Gamma \curvearrowright^{\alpha\times\beta} (X\times Y, \mu\times\nu)$ by $\gamma \cdot (x,y) := (\gamma_\alpha x, \gamma_\beta y)$. Let $\pi : X\times Y \to X$ be the X-projection. Then π is a factor map.



A factor $\Gamma \curvearrowright (Y, \nu)$ of $\Gamma \curvearrowright (X, \mu)$ can be identified with a Γ-invariant sub-σ-algebra of $\mathcal{B}(X)$, namely: $\mathcal{B}_Y := \pi^{-1} \mathcal{B}(Y)$. This $\mathcal{B}_Y$ is the σ-algebra of Borel π-invariant sets (for products, it's Borel sets that are unions of "columns"). Conversely, if $\mathcal{C} \subseteq \mathcal{B}(X)$ is a Γ-invariant sub-σ-algebra, then $\exists$ factor map $\pi : X \to (2^{\mathbb{N}})^\Gamma$ with some measure ν on $(2^{\mathbb{N}})^\Gamma$ s.t. $\mathcal{C} = \pi^{-1} \mathcal{B}((2^{\mathbb{N}})^\Gamma)$. Thus invariant sub-$\sigma$-algebras are called factor sub-σ-algebras or just factors.

Let $\mathbb{Z} \overset{T}{\curvearrowright} (X, \mu)$ and let $\mathbb{Z} \overset{S}{\curvearrowright} (Y, \nu)$ be a factor, corresponding to sub-σ-algebra $\mathcal{B}_Y$. To define what it means for this extension to be weakly mixing or compact, we look at each preimage X_y of an S-orbit $[y]_S$ and say that the restriction of T on X_y is weakly mixing or compact, but we want to do so uniformly. Since both notions are defined using inner product, we relativize the notion of inner product to $\mathcal{B}_Y$ as follows: $\forall f, g \in L^2(X, \mu)$

$$\langle f, g \rangle_Y := \mathbb{E}_\mu (fg \mid \mathcal{B}_Y) \text{ it's a function.}$$

conditional expectation

Define $L^2(X|Y) := \{ f \in L^2(X,\mu) : \|f\|_2^Y \in L^\infty(X,\mu) \}$.
Now we can define weakly mixing & compact for this new Hilbert module over $L^\infty(X,\mu)$.

Def. The extension $X \to Y$ is weakly mixing if $\forall f,g \in L^2(X|Y)$ with $\mathbb{E}(f|\mathcal{B}_Y) = 0 = \mathbb{E}(g|\mathcal{B}_Y)$,
$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \langle T^n f, g\rangle_Y = 0 \text{ in } L^2\text{-norm.}$$

It's harder to define compact extensions & we'll skip it.
What Furstenberg really proved is the following:

Theorem. Let $X \to Y$ be an extension of pmp $\mathbb{Z}$-actions.
(a) If Y is (MR) & this extension is either weakly mixing or compact, then X too is (MR).
(b) Dichotomy: If this extension is not weakly mixing, then $\exists$ nontrivial $X \to Y^+ \to Y$ s.t. $Y^+ \to Y$ is compact.

Thus, if by Zorn's lemma, we take a maximal (MR)

factor Y of X then Y would have to be equal to X
because otherwise, either $X \to Y$ weakly mixing
or $\exists\ X \to Y^{+} \to Y$ with $Y^{+} \to Y$ compact

and either case contradicts the maximality of Y
by (a). This "proves" Furstenberg MR theorem.
And the sequence (ordinal-length) one gets instead of Zorn:
$X_0 := \{\cdot\} \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_\omega \leftarrow X_{\omega+1} \leftarrow X_{\omega+2} \leftarrow \cdots$
$\cdots \leftarrow X_\alpha = X$. This is called a Furstenberg
tower. $\square$